Morse theory for group presentations

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EPSRC Centre for Topological Data Analysis

Durham University

Outline

1. The Andrews–Curtis conjecture

Motivation

Is every compact n-dimensional manifold homotopy equivalent to S^n if and only if it is homeomorphic with S^n?

H. Poincaré (1904)

The topological story

J.H.C. Whitehead,

- Simplicial spaces, nuclei and m-groups, Proc. London Math. Soc. 45 (1939) 243–327.
- On incidence matrices, nuclei and homotopy types, Ann. of Math. 42 (1941) 1197–1239.
- Combinatorial Homotopy I , Bull. Amer. Math. Soc, 55, (1949), 213–245.
- Combinatorial Homotopy II, Bull. Amer. Math. Soc., 55 (1949), 453–496.
- Simple homotopy types, Amer. J. Math. 72 (1950) 1–57.

Whitehead's simple homotopy theory

Let K, L be CW-complexes.

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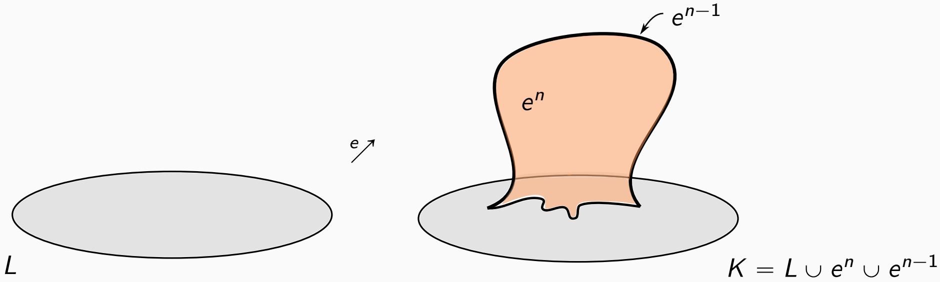

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- Collapse/expansion: $K \searrow L$ (or $L \nearrow K$) if there is a finite sequence of elementary collapses from K to L.
- n-deformation: $K \overset{n}{\nearrow\!\searrow} L$ if there is a sequence of CW-complexes $K = K_0, K_1, \dots K_r = L$ such that $K_i \overset{e}{\searrow} K_{i+1}$ or $K_i \overset{e}{\nearrow} K_{i+1}$ for each $0 \leqslant i \leqslant r-1$, and $\dim(K_i) \leqslant n$ for all $1 \leqslant i \leqslant r$.

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- Simple homotopy type: $K \nearrow\searrow L$ if $K \overset{n}{\nearrow\searrow} L$ for some $n \in \mathbb{N}$.

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- (C.T.C. Wall '66) If K, L are 2-complexes, then $K \nearrow\searrow L \Rightarrow K \overset{4}{\nearrow\searrow} L$

The algebraic story

- J.J. Andrews, M.L. Curtis, "Free groups and handlebodies" Proc. Amer. Math. Soc., **16** (1965) pp. 192–195.

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Conjecture [Andrews & Curtis, 1965]

Any balanced presentation $\mathcal{P} = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$ of the trivial group can be transformed into $\langle x_1, \ldots, x_n \mid x_1, \ldots, x_n \rangle$ by a finite sequence of the following transformations:

- replace some relator r_i by r_i^{-1};
- replace some relator r_i by $r_i r_j$ for some $j \neq i$;
- replace some relator r_i by a conjugate $w r_i w^{-1}$ for some w in the free group $F(x_1, x_2, \ldots, x_n)$.

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Consequences:

- If a homotopy 4-sphere has a 2-spine (and the AC-conjecture is true), then it is a 4-sphere.
- If the AC-conjecture is true, and the 3-dimensional Poincaré conjecture is false, then a counterexample exists in 4-space.

The algebraic story

Theorem [Tietze, 1908]

Any finite presentation $\mathcal{P} = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$ of a group G can be transformed into any other presentation of the same group by a finite sequence of the following **operations**:

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4. add a generator x_{n+1} and a relator r_{m+1} that coincides with x_{n+1}, or the inverse of this operation;
5. add a relator 1, or the inverse of this operation.

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Q^{**} (operations 1–4), Q^{*} (operations 1–3):

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Group presentations $\longleftrightarrow$ CW-complexes of dim 2

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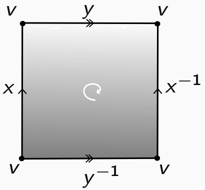
$$\mathcal{P} \longrightarrow K_{\mathcal{P}}$$

$$\langle x, y \mid xyx^{-1}y^{-1} \rangle \longrightarrow$$



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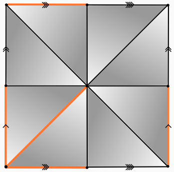
$\mathcal{P}_K \longleftarrow K$

$\langle x_1 \dots, x_9 \mid x_7,\ x_6^{-1}x_7^{-1}x_3,\ x_3^{-1}x_9,\ x_1^{-1}x_3^{-1}x_{10},\ x_4^{-1}x_5,\ x_5x_6^{-1},\ x_7,\ x_6x_7^{-1}x_8,\ x_8^{-1}x_9,\ x_6x_9^{-1} \rangle \longleftarrow$



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(Geometric) Conjecture [Andrews–Curtis, 1965]

Any contractible 2-complex 3-deforms to a point.

(Algebraic) Conjecture [Andrews–Curtis, 1965]

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The Andrews–Curtis conjecture today

- The conjecture is **true** for some classes of complexes, such as the *standard spines* (Gillman–Rolfsen 1983, Perelman 2002) and *quasi-constructibles* (Barmak, 2009).

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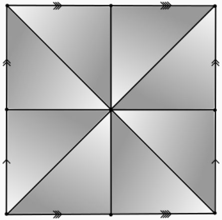
- There are some algorithms based on the exploration and exhibition of possible Q^*-transformations that work in **small** examples (Miasnikov et.al. 1999, 2002, Havas and Ramsay 2003, Bowman and McCaul 2006, Krawiec and Swan 2016).
- There is a list of potential counterexamples.
  - $\mathcal{P} = \langle x, y \mid xyx = yxy,\ x^n = y^{n+1} \rangle,\ n \geqslant 2$ [Akbulut & Kirby, 1985]
  - $\mathcal{P} = \langle x, y \mid x^{-1}y^n x = y^{n+1},\ x = y^{-1}xyx^{-1} \rangle,\ n \geqslant 2$ [Miller & Schupp, 1999]
  - $\mathcal{P} = \langle x, y \mid x = [x^m, y^n], y = [y^p, x^q] \rangle\ n, m, p, q \in \mathbb{Z}$ [Gordon, 1984]

2. Discrete Morse theory

Let K be a **regular** CW-complex.



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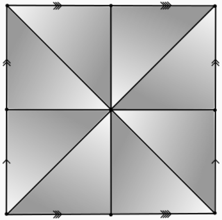
- A map $f : K \to \mathbb{R}$ is a discrete Morse function if for every cell $\boldsymbol{e^n}$ in K:

  $\#\{\boldsymbol{e^n} > e^{n-1} : f(\boldsymbol{e^n}) \leqslant f(e^{n-1})\} \leqslant 1$ and $\#\{\boldsymbol{e^n} < e^{n+1} : f(\boldsymbol{e^n}) \geqslant f(e^{n+1})\} \leqslant 1.$



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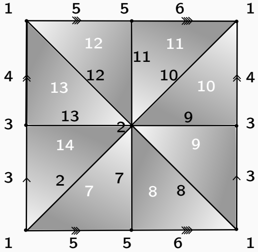

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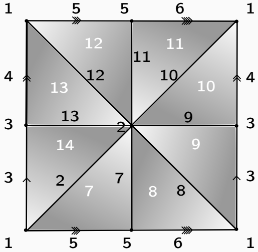
- An n-cell $e^n \in K$ is a critical cell of index n if the values of f in every face and coface of e^n increase with dimension.



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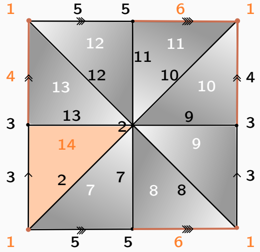

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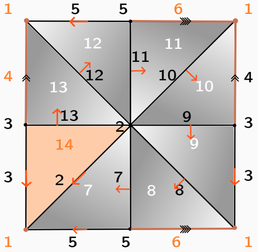


Morse theory for cell complexes

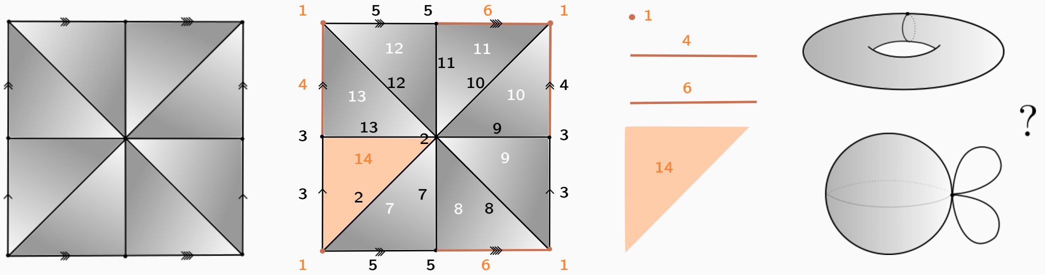
Theorem [Forman, 1995]

Let K be a regular CW-complex and let $f : K \to \mathbb{R}$ be a discrete Morse function. For every $c \in \mathbb{R}$, consider the *level subcomplex $K(c)$ of K*, that is, the subcomplex of closed cells $\bar{e}$ of K such that $f(e) \leqslant c$ in $\mathbb{R}$. Let $a < b$ be real numbers.

(a) If every cell $e \in K$ such that $f(e) \in (a, b]$ is **not critical**, then $K(b) \searrow K(a)$.

(b) If $e^k \in K$ is the only **critical** cell with $f(e^k) \in (a, b]$, then there is a continuous map $\varphi : \partial D^k \to K(a)$ such that $K(b) \simeq K(a) \cup_\varphi D^k$.

(c) K is homotopy equivalent to a CW-complex $K_{\mathcal{M}}$ with exactly one cell of dimension k for every critical cell of index k.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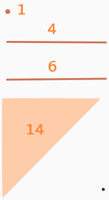
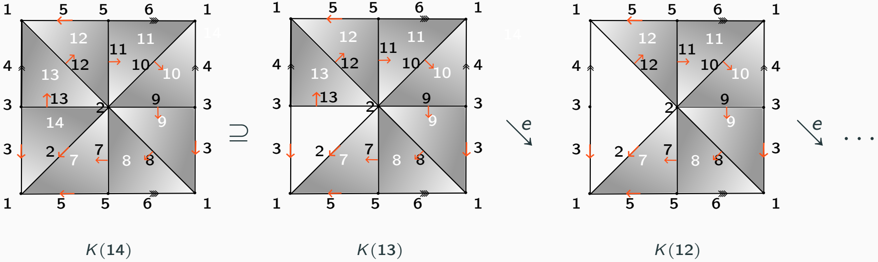
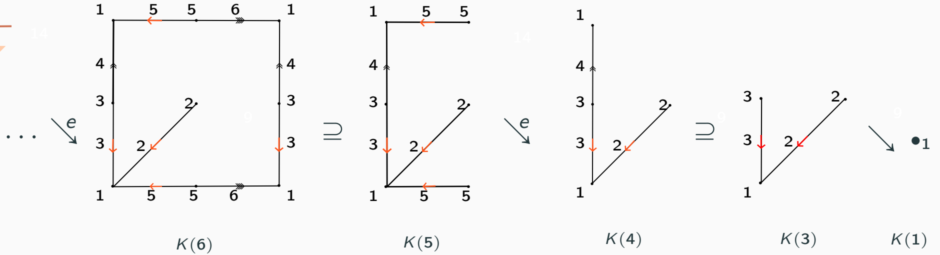
Morse theory and collapses

Level subcomplex

$$K(c) = \bigcup_{\substack{e \in K \\ f(e) \leqslant c}} \bar{e}$$







Morse theory and simple homotopy reconstruction

Goals:

Given K a regular CW-complex of dimension n and a discrete Morse function $f : K \to \mathbb{R}$, we aim to:

- (re)construct the Morse complex $K_{\mathcal{M}}$,
- recover information about the simple homotopy type of K and moreover its $(n+1)-$deformation class.

Lema [F.]

Let K be a regular CW-complex.

Then, $f : K \to \mathbb{R}$ is a discrete Morse function with critical cells C if and only if there exist a sequence of subcomplexes of K

$$K_0 \subseteq L_1 \subseteq K_1 \cdots \subseteq K_{N-1} \subseteq L_{N-1} \subseteq K_N = K$$

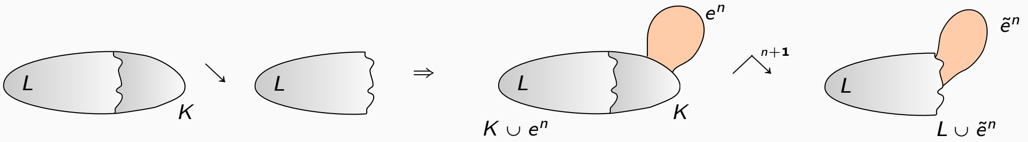
such that $K_j \searrow L_j$ for all $1 \leqslant j \leqslant N$ and the set of cells of K that was not collapsed in any of the collapses $K_j \searrow L_j$ is equal to C.

Lema

Let K be a CW-complex of dimension $\leqslant n$. Let $\varphi : \partial D^n \to K$ be the attaching map of an n-cell e^n. If $K \searrow L$, then

$$K \cup e^n \nearrow\!\!\searrow^{n+1} L \cup \tilde{e}^n$$

where the attaching map $\tilde{\varphi} : \partial D^n \to L$ of $\tilde{e}^n$ is defined as $\tilde{\varphi} = r\varphi$ with $r : K \to L$ the canonical strong deformation retract induced by the collapse $K \searrow L$.



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Definition

We say that there is an **internal collapse** from $K \cup e^n$ to $L \cup \tilde{e}^n$.

Internal collapses

Proposition [F.]

Let K be a CW-complex on dimension n. Let

$$\varnothing = K_{-1} \subseteq L_0 \subseteq K_0 \subseteq L_1 \subseteq K_1 \cdots \subseteq L_N \subseteq K_N = K$$

be a sequence of subcomplexes of L such that $K_j \searrow L_j$ for all $j = 0, 1, \ldots N$. If $L_j = K_{j-1} \cup \bigcup_{i=1}^{d_j} e_i^j$, then

$$K \nearrow^{n+1} L_0 \cup \bigcup_{j=0}^{N} \bigcup_{i=1}^{d_j} \tilde{e}_i^j.^*$$

*Here, the attaching maps of the cells $\tilde{e}_i^j$ can be explicitly reconstructed from the internal collapses.

Morse theory and Whitehead deformations

Theorem [F.]

Let K be a regular CW-complex of dimension n and let $f : K \to \mathbb{R}$ be discrete Morse function. Then, f induces a sequence of CW-subcomplexes of K

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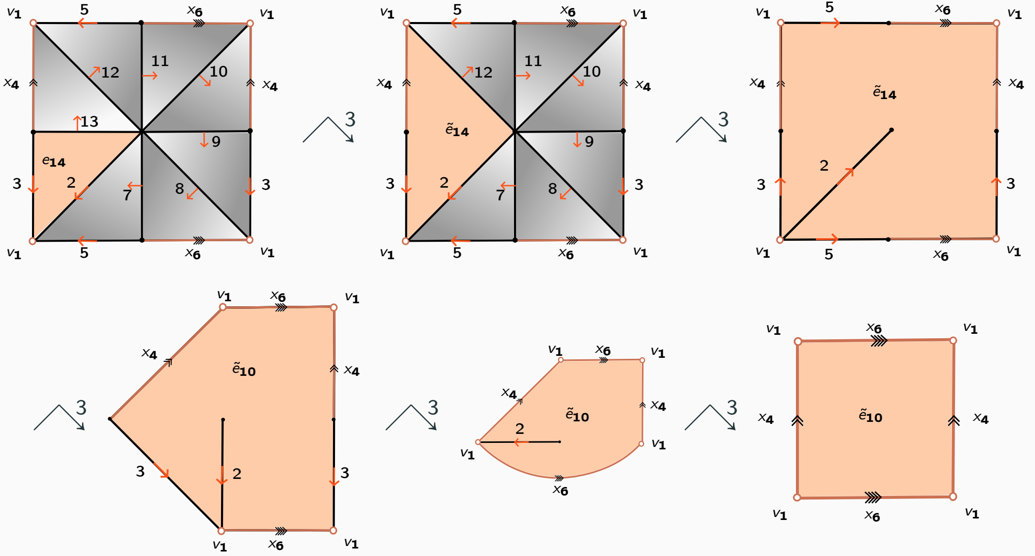
such that $K_j \searrow L_j$ for all $1 \leqslant j \leqslant N$ and $L_j = K_{j-1} \cup \bigcup_{i=1}^{d_j} e_i^j$ with $\{e_i^j : 0 \leqslant j \leqslant N, 1 \leqslant i \leqslant d_j\}$ the set of critical cells of f. Moreover,

$$K \overset{n+1}{\diagup\!\!\searrow} L_0 \cup \bigcup_{j=1}^{N} \bigcup_{i=1}^{d_j} \tilde{e}_i^j = K_{\mathcal{M}}.$$

Example



3. Morse theory for group presentations

Pipeline

Group presentations $\mathcal{P}$

Pipeline

Group presentations	$\mathcal{P}$
	$\rightsquigarrow$
CW-complexes	$K_{\mathcal{P}}$

Pipeline

Group presentations $\mathcal{P}$

$$\rightsquigarrow$$

CW-complexes $K'_{\mathcal{P}}$

Pipeline

Group presentations $\mathcal{P}$

$\rightsquigarrow$

CW-complexes $K'_{\mathcal{P}} + f$

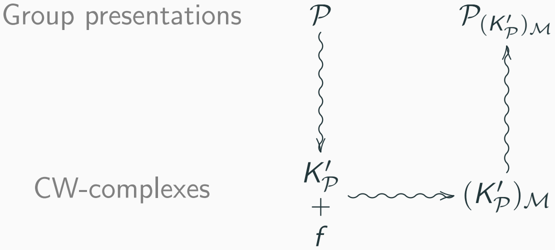
Pipeline

Group presentations $\mathcal{P}$

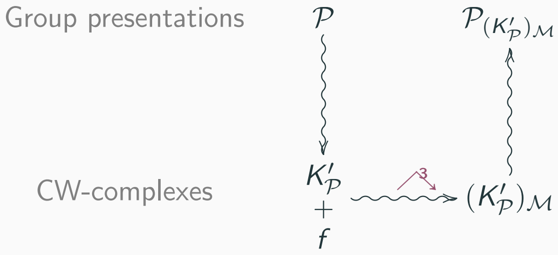
$$\Big\downarrow$$

CW-complexes $K'_{\mathcal{P}} + f \rightsquigarrow (K'_{\mathcal{P}})_{\mathcal{M}}$

Pipeline


Group presentations
$\mathcal{P}$
$\mathcal{P}_{(K'_{\mathcal{P}})_{\mathcal{M}}}$
CW-complexes
$K'_{\mathcal{P}}$
$+$
f
$(K'_{\mathcal{P}})_{\mathcal{M}}$

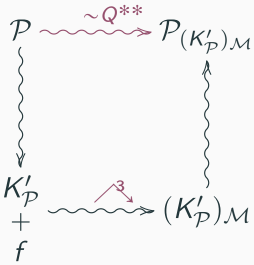
Pipeline


Group presentations
$\mathcal{P}$
$\mathcal{P}_{(K'_{\mathcal{P}})_{\mathcal{M}}}$
CW-complexes
$K'_{\mathcal{P}} + f$
3
$(K'_{\mathcal{P}})_{\mathcal{M}}$

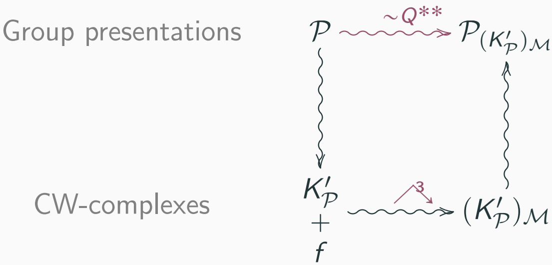
Pipeline

Group presentations

CW-complexes



Pipeline



We have an algorithmic description of $\mathcal{P}_{(K'_{\mathcal{P}})_{\mathcal{M}}}$ from $\mathcal{P}$.

Example

$$\mathcal{P} = \langle x, y \mid x^2, xy^{-2} \rangle$$

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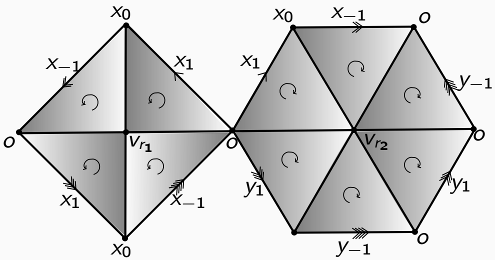
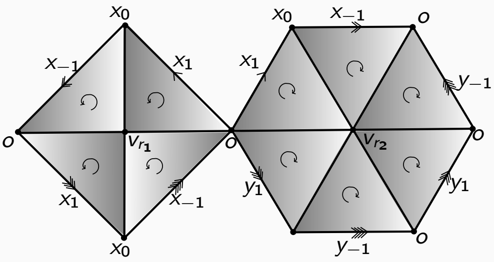
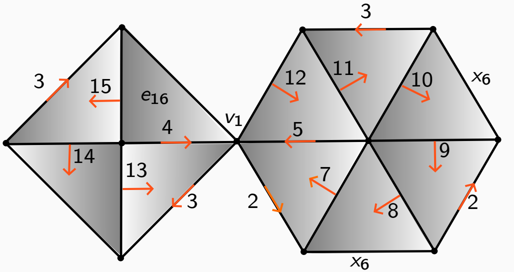


$K'_{\mathcal{P}}$

$$\mathcal{P} = \langle x, y \mid x^2, xy^{-2} \rangle$$



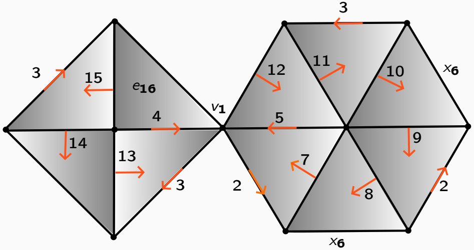
$K'_{\mathcal{P}}$



$+ \quad f : K'_{\mathcal{P}} \to \mathbb{R}$ discrete Morse function

Example: $\mathcal{P} = \langle x, y \mid x^2, xy^{-2} \rangle$

$$\mathcal{Q}_0 = \langle x_2, x_3, x_4, x_5, x_6, \ldots, x_{15} \mid x_2, x_3, x_4, x_5, x_7 x_2^{-1} x_5^{-1},\ x_6^{-1} x_7^{-1} x_8,\ x_8^{-1} x_9 x_2^{-1},\ x_6^{-1} x_9^{-1} x_{10},\ x_{10}^{-1} x_{11} x_3,\ x_{12} x_{11}^{-1} x_5,\ x_{13} x_3 x_4^{-1},\ x_{12} x_{13}^{-1} x_{14},\ x_{14}^{-1} x_{15} x_3,\ x_4 x_{12} x_{15}^{-1} \rangle$$



Example: $\mathcal{P} = \langle x, y \mid x^2, xy^{-2} \rangle$

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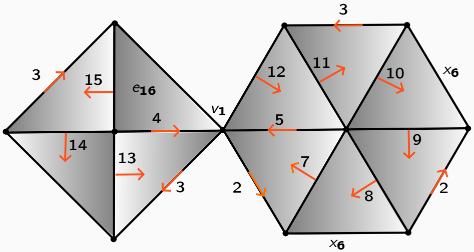
$$x_7 x_2^{-1} x_5^{-1},\; x_6^{-1} x_7^{-1} x_8,\; x_8^{-1} x_9 x_2^{-1},\; x_6^{-1} x_9^{-1} x_{10},\; x_{10}^{-1} x_{11} x_3,\; x_{12} x_{11}^{-1} x_5,\; x_{13} x_3 x_4^{-1},\; x_{12} x_{13}^{-1} x_{14},\; x_{14}^{-1} x_{15} x_3,\; x_4 x_{12} x_{15}^{-1} \rangle$$

$$\mathcal{Q}_1 = \langle x_6 \ldots, x_{15} \mid x_7,\; x_6^{-1} x_7^{-1} x_8,\; x_8^{-1} x_9,\; x_6^{-1} x_9^{-1} x_{10},\; x_{10}^{-1} x_{11},\; x_{11} x_{12}^{-1},\; x_{13},\; x_{12} x_{13}^{-1} x_{14},\; x_{14}^{-1} x_{15},\; x_{12} x_{15}^{-1} \rangle$$



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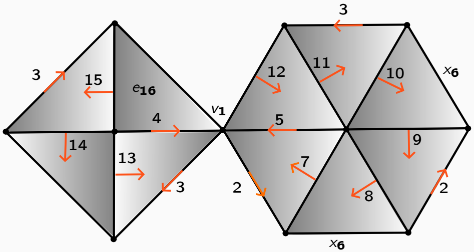
$$\mathcal{Q}_2 = \langle x_6, \ldots, x_{14} \mid x_7,\ x_6^{-1}x_7^{-1}x_8,\ x_8^{-1}x_9,\ x_6^{-1}x_9^{-1}x_{10},\ x_{10}^{-1}x_{11},\ x_{11}x_{12}^{-1},\ x_{13},\ x_{12}x_{13}^{-1}x_{14},\ x_{12}x_{14}^{-1} \rangle$$



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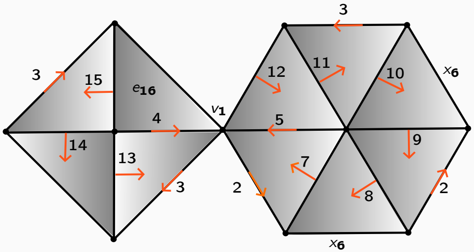

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$$\mathcal{Q}_4 = \langle x_6, \ldots, x_{12} \mid x_7,\ x_6^{-1}x_7^{-1}x_8,\ x_8^{-1}x_9,\ x_6^{-1}x_9^{-1}x_{10},\ x_{10}^{-1}x_{11},\ x_{11}x_{12}^{-1},\ x_{12}^2 \rangle$$

$$\mathcal{Q}_5 = \langle x_6, \ldots, x_{11} \mid x_7,\ x_6^{-1}x_7^{-1}x_8,\ x_8^{-1}x_9,\ x_6^{-1}x_9^{-1}x_{10},\ x_{10}^{-1}x_{11},\ x_{11}^2 \rangle$$

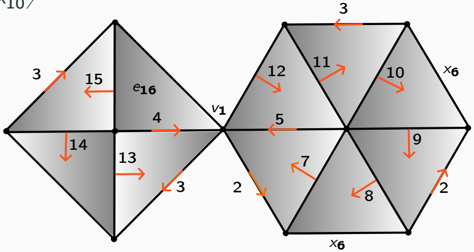
$$\mathcal{Q}_6 = \langle x_6, \ldots, x_{10} \mid x_7,\ x_6^{-1}x_7^{-1}x_8,\ x_8^{-1}x_9,\ x_6^{-1}x_9^{-1}x_{10},\ x_{10}^2 \rangle$$

$$\mathcal{Q}_7 = \langle x_6, \ldots, x_9 \mid x_7,\ x_6^{-1}x_7^{-1}x_8,\ x_8^{-1}x_9,\ (x_9x_6)^2 \rangle$$

$$\mathcal{Q}_8 = \langle x_6, x_7, x_8 \mid x_7,\ x_6^{-1}x_7^{-1}x_8,\ (x_8x_6)^2 \rangle$$

$$\mathcal{Q}_9 = \langle x_6, x_7 \mid x_7, (x_7x_6^2)^2 \rangle$$

$$\mathcal{Q}_{10} = \langle x_6 \mid x_6^4 \rangle \sim_{Q**} \mathcal{P}$$



4. Applications to the Andrews–Curtis conjecture

Potential counterexamples

Theorem [F.]

The following balanced presentations of the trivial group satisfies the Andrews–Curtis conjecture:

- $\mathcal{P} = \langle x, y \mid xyx = yxy,\ x^2 = y^3 \rangle^{\dagger}$ [Akbulut & Kirby, 1985]
- $\mathcal{P} = \langle x, y \mid x^{-1}y^3x = y^4,\ x = y^{-1}xyx^{-1} \rangle$ [Miller & Schupp, 1999]
- $\mathcal{P} = \langle x, y \mid x = [x^{-1}, y^{-1}], y = [y^{-1}, x^q] \rangle, \forall q \in \mathbb{N}$ [Gordon, 1984]

† First proved by Miasnikov in 2003 using genetic algorithms.

The search for Morse functions

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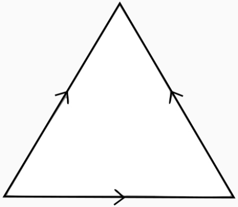

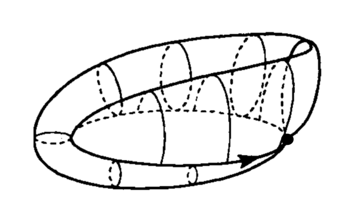

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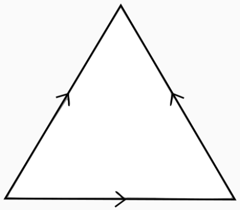

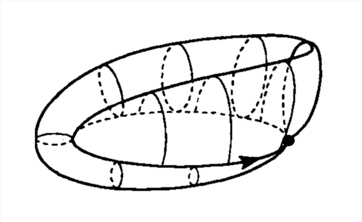

- There is standard a **greedy algorithm** $\mathcal{A}$ to trivialize a presentation using Q^*-transformations (Havas, Kenne, Richardson & Roberts, 1984).

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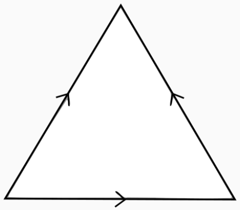
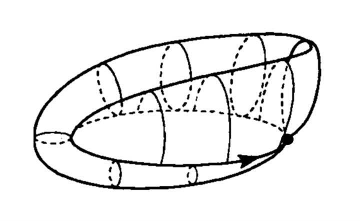

- There is standard a **greedy algorithm** $\mathcal{A}$ to trivialize a presentation using Q^*-transformations (Havas, Kenne, Richardson & Roberts, 1984).
  Given a presentation $\mathcal{P}$, we search for Morse functions such that $\mathcal{P}_{\mathcal{K}'_{\mathcal{P}}}$ is **trivializable by** $\mathcal{A}$.

MOSTRAR CODIGO!!!

Geometric potential counterexamples

We have proved that the following contractible 2-complexes 3-deforms to a point:

ACA VAN LOS DIBUJOS DE LOS CW ASOCIADOS A LAS PRESENTACIONES (SIN SUDIVIDIR)

References

- *PhD Thesis:* X. F., *Combinatorial methods and algorithms in low-dimensional topology and the Andrews-Curtis conjecture*, University of Buenos Aires, 2017.
- *Preprint:* X. F., *Morse theory for group presentations*, arXiv:1912.00115, 2021.
- *Code:*
  - X. F., SageMath Module, `https://github.com/ximenafernandez/Finite-Topological-Spaces`
  - X. F., Kevin Piterman & Ivan Sadofschi Costa, GAP Package, `https://github.com/isadofschi/posets`

Work in progress: Computation of persistent fundamental group of point clouds.

✉ `ximena.l.fernandez@durham.ac.uk`

Thanks for your attention!