3-DEFORMATIONS OF 2-COMPLEXES AND MORSE THEORY

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ABSTRACT. We introduce novel combinatorial methods to study 3-deformations of CW-complexes of dimension 2 or, equivalently, Q^{**}-transformations of group presentations. Our procedures are based on a new version of Discrete Morse Theory for n-deformations. We apply these techniques to show that some known potential counterexamples to the Andrews-Curtis conjecture do satisfy the conjecture.

The Andrews Curtis conjecture is an extensively studied open problem in two-dimensional geometric topology, with roots in Whitehead's simple homotopy theory [29, 30] and combinatorial group theory. This conjecture is closely related to other relevant problems in algebraic topology, such as Whitehead asphericity conjecture [28], Zeeman conjecture [32] and the Poincaré conjecture (now a theorem). It states that if K is a (finite) contractible CW-complex of dimension 2, then it 3-deforms to a point, i.e. it can be transformed into a point by a sequence of expansions and collapses in which the dimension of the complexes involved is not greater than 3. Although the conjecture is known to be true for some classes of complexes (such as the *standard spines* [24] and the *quasi-constructible complexes* [3]), it still remains open for general 2-complexes.

The conjecture was originally stated in terms of group presentations [1]. Namely, given a balanced presentation of the trivial group $\mathcal{P} = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$, the conjecture states that $\mathcal{P}$ can be transformed into the empty presentation $\langle \mid \rangle$ by a finite sequence of the following operations, which we call *AC-transformations*:

(1) replace some r_i by r_i^{-1};
(2) replace some r_i by $r_i r_j$, $j \neq i$;
(3) replace some r_i by $w r_i w^{-1}$ (where w is any word in the generators);
(4) introduce a new generator y and the relator y, or the inverse of this operation.

Computational approaches have shown to be limited by the exponential complexity of the algorithms (see [6, 7, 18, 20, 21, 22]). In this article, we present new methods which combine topological and combinatorial tools, that allow the computational exploration of presentations which are Andrews-Curtis equivalent from a given one. These alternative techniques to explore the AC-transformations, based on Morse Theory, enable us to analyze some of the potential counterexamples to the conjecture.

We will work simultaneously with both formulations of the conjecture - whose equivalence was first noticed by the anonymous referee of the foundational article [1]-. In the same way as the collapses and expansions determine classes of (simple) homotopy of spaces, the transformations (1)-(4) define classes of group presentations, the *AC-classes*. Moreover, there is a correspondence between 3-deformation classes of complexes of dimension 2 and AC-classes of group presentations. There is a standard way to associate a presentation $\mathcal{P}_K$ to any 2-complex K and, conversely, a 2-complex $K_{\mathcal{P}}$ to every presentation $\mathcal{P}$. See

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[16, Ch. I, Sec. 1.3, 1.4]. We describe briefly this procedure in Section 1. Given balanced group presentations $\mathcal{P}, \mathcal{Q}$, if $\mathcal{P}$ can be transformed into $\mathcal{Q}$ by a finite sequence of AC-transformations, we say that $\mathcal{P}$ is *AC-equivalent* to $\mathcal{Q}$, and we denote $\mathcal{P} \sim_{AC} \mathcal{Q}$. It can be shown that $\mathcal{P} \sim_{AC} \mathcal{Q}$ if and only if $K_{\mathcal{P}}$ 3-deforms to $K_{\mathcal{Q}}$ (denoted by $K_{\mathcal{P}} \nearrow\searrow^3 K_{\mathcal{Q}}$). Similarly, if K, L are CW-complexes of dimension 2, $K \nearrow\searrow^3 L$ if and only if $\mathcal{P}_K \sim_{AC} \mathcal{P}_L$. Moreover, $\mathcal{P} \sim_{AC} \mathcal{P}_{K_{\mathcal{P}}}$ and $K \nearrow\searrow^3 K_{\mathcal{P}_K}$. Balanced presentations of the trivial group correspond to contractible complexes.

In [16], the Andrews-Curtis conjecture is stated in terms of Q^{**}-transformations, which allow an additional elementary operation:

(5) in the relators, replace x_i throughout by x_i^{-1} or $x_i x_j$ or $x_j x_i$, with $j \neq i$.

If a finite group presentation $\mathcal{P}$ can be transformed into another one $\mathcal{Q}$ through a sequence of operations (1) to (5), it is said that $\mathcal{P}$ is *Q^{**}-equivalent* to $\mathcal{Q}$, and denoted by $\mathcal{P} \sim_{Q^{**}} \mathcal{Q}$. By Nielsen's Theorem, operation (5) is not necessary for balanced presentations of the trivial group; i.e., $\mathcal{P} \sim_{Q^{**}} \langle \mid \rangle$ if and only if $\mathcal{P} \sim_{AC} \langle \mid \rangle$.

Discrete Morse theory provides combinatorial tools to describe simpler cell decompositions of a given CW-complex (up to homotopy equivalence). In this article we extend the scope of this theory to make it appropriate for handling 3-deformations of 2-complexes and the Andrews-Curtis conjecture. Concretely, given a regular n-complex K and a discrete Morse function on it, we construct an explicit, algorithmically computable cell decomposition of a complex that $(n+1)$-deforms to K. Our goal is to generate a method to obtain new presentations AC-equivalent to a given one without requiring to specify the exhaustive list of movements to transform one into the other, since it could be out of reach (see [7]).

Given $\mathcal{P}$ a finite group presentation, denote by $K'_{\mathcal{P}}$ the barycentric subdivision of $K_{\mathcal{P}}$. We define $X_{\mathcal{P}}$, the *presentation poset* of $\mathcal{P}$, as the *face poset* $\mathcal{X}(K'_{\mathcal{P}})$ of $K'_{\mathcal{P}}$, that is, the poset of cells of $K'_{\mathcal{P}}$ ordered by inclusion. We will introduce some combinatorial techniques to study the AC-class of a given presentation $\mathcal{P}$, by means of its presentation poset $X_{\mathcal{P}}$. In Section 1, we define a presentation $\mathcal{P}'$ associated to a subdiagram of the Hasse diagram of $X_{\mathcal{P}}$ and characterize a class of subdiagrams such that $\mathcal{P}'$ results AC-equivalent to $\mathcal{P}$. In Section 2, we make the reformulation of discrete Morse theory in terms of n-deformations, and associate to each acyclic matching in the Hasse diagram of $X_{\mathcal{P}}$ a presentation $\mathcal{Q}$ such that $\mathcal{P} \sim_{AC} \mathcal{Q}$. In Section 3, we present some applications of our methods to investigate potential counterexamples to the conjecture (see [17, Sec. 4.2.]). The results of this article are part of the author's PhD Thesis [11].

Independently, Brendel, Dlotko, Ellis, Juda and Mrozek presented in [5] an algorithm to describe a presentation of the fundamental group of a regular CW-complex, using Forman's combinatorial version of Morse theory. They applied it in a classification problem of prime knots, and to compute the fundamental group of point clouds.

1. Presentations and posets

We recall briefly the main concepts in simple homotopy theory. We refer the reader to [10, 16] for a more complete exposition. All the CW-complexes in this article will be finite and connected. Given a CW-complex K and a subcomplex L of K, we say that K *elementary collapses* to L, and we denote it by $K \searrow^e L$, if $K = L \cup e^{n-1} \cup e^n$ with $e^{n-1}, e^n \notin L$ and there exists a map $\psi : D^n \to K$ such that ψ is the characteristic map of e^n, $\psi|_{\overline{\partial D^n \smallsetminus D^{n-1}}}$ is the characteristic map of e^{n-1} and $\psi(D^{n-1}) \subseteq L^{n-1}$. In general, K *collapses* to another L (or L *expands* to K) if there is a finite sequence of elementary collapses from K to L. We denote it by $K \searrow L$ (resp. $L \nearrow K$), We say that a CW-complex K *n-deforms* to L, and we denote it by $K \nearrow\!\!\!\searrow^{n} L$, if there is a sequence of CW-complexes $K = K_0, K_1, \cdots K_r = L$ such that for each $0 \leq i \leq r-1$, $K_i \searrow^{e} K_{i+1}$ or $K_i \,{}^{e}\!\!\nearrow K_{i+1}$, and $\dim(K_i) \leq n$ for all $1 \leq i \leq r$. For every $0 \leq i \leq r-1$, there is a homotopy equivalence $f_i : K_i \to K_{i+1}$ which is an inclusion or a retraction depending on whether $K_i \,{}^{e}\!\!\nearrow K_{i+1}$ or $K_{i+1} \searrow^{e} K_i$ respectively. In that case, K and L are then related by a *deformation* $f : K \to L$ defined to be the composition of the retractions and inclusions as above, i.e, $f = f_{r-1} \cdots f_1 f_0$. Notice that if $K \nearrow\!\!\!\searrow^{n} L$, then K and L are homotopically equivalent.

If K is a finite CW-complex of dimension 2 and T is a spanning tree of $K^{(1)}$ (the 1-skeleton of K), then $K \nearrow\!\!\!\searrow^{3} K/T$ and $\mathcal{P}_K$ is defined as a presentation of the fundamental group of K/T. Different choices of spanning trees result in AC-equivalent presentations (see [16, Ch. I, Sec. 2.3.][31]). It is important to note that $\mathcal{P}_K$ has not only the information about the fundamental group of K, but also about its 3-deformation class.

In this section we extend the idea of obtaining new spaces which 3-deforms to a given one, by taking an appropriate quotient space of it. We also interpret these methods in the context of subdiagrams of posets. Recall that a poset X can be represented as a directed graph, its *Hasse diagram* (denoted by $\mathcal{H}(X)$) whose vertices are the elements of X, and whose edges are the pairs (x, y) such that $x \prec y$, i.e., $x < y$ and there is no $z \in X$ satisfying $x < z < y$. A *chain* in X is a totally ordered subposet of X. The *height* $ht(X)$ of a finite poset X is one less than the maximum number of elements in a chain in X. The *height* $ht(x)$ of a point x in X is the height of the subposet of X given by the elements $y \in X$ such that $y \leq x$. All the posets in this article will be finite and connected (that is, its Hasse diagram will be a finite connected graph).

Definition 1.1. Let X be a poset of height 2 and let C be a subgraph (also called *subdiagram*) of $\mathcal{H}(X)$. We say that (x, y) is an *extremal pair* of X if $x < y$, $ht(x) = 0$ and $ht(y) = 2$. For every extremal pair (x, y) of X, if there exists any chain $c_{x,y} : x \prec z \prec y$ such that at least one of its edges is not in C, fix a *preferred* one. We associate to (X, C) a group presentation $\mathcal{P}_{X,C}$ whose generators are the edges of $\mathcal{H}(X)$ not belonging to C, and whose relators are induced by the *digons* (i.e. pairs of monotonic edge paths that only meet in the extremal points) containing a preferred chain. If the preferred chain $x < z < y$ and the chain $x < z' < y$ form a digon, then the induced relator follows from the equality of the words read off from these chains.

To every finite poset X one can associate a finite simplicial complex $\mathcal{K}(X)$, the *order complex* of X, whose vertices are the elements of X, and whose simplices are the nonempty chains of X. In Theorems 1.3 and 1.7 we will show that, for conveniently chosen subdiagrams C, the presentation $\mathcal{P}_{X,C}$ associated to a poset X is AC-equivalent to the classical presentation $\mathcal{P}_{\mathcal{K}(X)}$ of the associated complex, independently of the choice of the preferred chains.

Example 1.2. Consider the two different subdiagrams, identified with dotted lines, of the finite poset of Figure 1 whose order complex is a triangulation of the Projective Plane. Since there are only two chains between every pair of points of heights 0 and 2 respectively, any choice of preferred chains will result in the same presentation. C_1 is a spanning tree of $\mathcal{H}(X)$ and C_2 has all the vertices of X and satisfies that $\mathcal{K}(C_2)$ is collapsible. We get $\mathcal{P}_{X,C_1} = \langle x_1, x_2, \ldots, x_{12} \mid x_8 = 1,\ x_1x_6 = 1,\ x_1x_7x_2^{-1} = 1,\ x_2x_9x_5^{-1} = 1,\ x_{10} = 1,\ x_6x_{12}^{-1} = 1,\ x_7x_{11}^{-1} = 1,\ x_5 = 1, x_8x_{10}^{-1} = 1,\ x_{12} = 1,\ x_4x_{11}, = 1\ x_9 = 1 \rangle$ and $\mathcal{P}_{X,C_2} =$

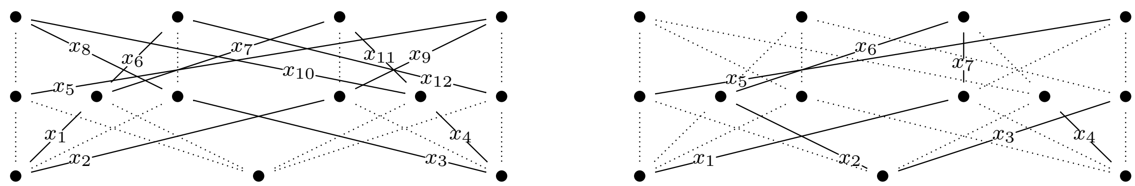

FIGURE 1. Subdiagrams C_1 and C_2 of $\mathcal{H}(X)$.

$\langle x_1, x_2, \ldots, x_7 \mid x_1x_7 = x_6,\ x_1 = x_5,\ x_3 = 1,\ x_2 = x_3,\ x_2x_6 = 1,\ x_5 = x_3,\ x_4 = 1,\ x_4 = x_7\rangle$. Note that $\mathcal{P}_{X,C_1}$ and $\mathcal{P}_{X,C_2}$ are presentations of $\mathbb{Z}_2$.

Theorem 1.3. *Let X be a finite poset and T be a spanning tree of $\mathcal{H}(X)$. Then, $\mathcal{P}_{X,T} \sim_{AC} \mathcal{P}_{\mathcal{K}(X)}$.*

Proof. Let T be a spanning tree of $\mathcal{H}(X)$. Note that we can think of T as a spanning tree of the 1-skeleton of $\mathcal{K}(X)$, since there is an inclusion $\mathcal{H}(X) \subseteq \mathcal{K}(X)^{(1)}$ into the 1-skeleton, viewed as undirected graphs. Fix an orientation of the 1-cells inherited from the partial order in X. Let $\mathcal{P}_{\mathcal{K}(X),T}$ be the presentation of the fundamental group of $\mathcal{K}(X)/T$ obtained by the previously selected orientation of the 1-cells. We verify that $\mathcal{P}_{\mathcal{K}(X),T}$ can be transformed into $\mathcal{P}_{X,T}$ through AC-transformations. For every extremal pair (x, y) of X, fix a preferred chain $c^0_{xy} : x < z < y$ such that at most one of the edges (x, z) and (z, y) is in T (note that such a chain exists for every extremal pair because T has no cycles). Label $e_1, \ldots, e_r$ the edges in $\mathcal{H}(X) \smallsetminus T$. Call w^0_{xy} the word associated to c^0_{xy} in the free group $F(e_1, \ldots, e_r)$ generated by $e_1, \ldots, e_r$. For every other chain c^i_{xy} between x, y $(1 \leq i \leq n_{xy})$, call w^i_{xy} the associated word. Thus, the generators of $\mathcal{P}_{X,T}$ are $e_1, \ldots, e_r$, and its relators are $(w^i_{xy})^{-1}w^0_{xy}$ for every extremal pair (x, y) of X and for every chain $c^i_{xy} \neq c^0_{xy}$. Summarizing,

$$\mathcal{P}_{X,T} = \langle e_1, \ldots, e_r \mid \{(w^i_{xy})^{-1}w^0_{xy} : (x, y) \text{ extremal pair of } X, 1 \leq i \leq n_{xy}\}\rangle.$$

On the other hand, if we denote e_{xy} the edge in $\mathcal{K}(X)^{(1)}$ associated to the extremal pair (x, y), then

$$\mathcal{P}_{\mathcal{K}(X),T} = \langle e_1, \ldots, e_r, e_{xy} \mid \{e^{-1}_{xy}w^0_{xy},\ (w^i_{xy})^{-1}e_{xy} : (x, y) \text{ extremal pair of } X, 1 \leq i \leq n_{xy}\}\rangle.$$

For every extremal pair (x, y) call $r_i = (w^i_{xy})^{-1}e_{xy}$, $r_0 = e^{-1}_{xy}w^0_{xy}$. In $\mathcal{P}_{\mathcal{K}(X),T}$, replace r_i by $r_ir_0 = (w^i_{xy})^{-1}w^0_{xy}$, and finally eliminate the generator e_{xy} together with the relator $r_0 = w^0_{xy}e_{xy}$. □

Corollary 1.4. *Let X be a finite poset of height 2. Then, different choices of a spanning tree in $\mathcal{H}(X)$ and preferred chains between related maximal and minimal elements of X give rise to AC-equivalent presentations of $\pi_1(X)$.*

Proof. The proof of Theorem 1.3 shows that for any choice of preferred chains, if we fix T a spanning tree in $\mathcal{H}(X)$, then $\mathcal{P}_{X,T} \sim_{AC} \mathcal{P}_{\mathcal{K}(X)}$. Since the AC-equivalence class of $\mathcal{P}_{\mathcal{K}(X)}$ does not depend on the spanning tree T the same holds for $\mathcal{P}_{X,T}$. □

We now extend the class of good choices of subdiagrams of a poset to *collapsible subdiagrams* containing a spanning tree (that is, subdiagrams containing all the vertices such that its order complex is collapsible).

Proposition 1.5. *Let K be a CW-complex of dimension 2. Let $A \leq K$ be a collapsible subcomplex containing all vertices of K. Then $\mathcal{P}_{K/A} \sim_{AC} \mathcal{P}_K$.*

Proof. Since A is a collapsible subcomplex of K, there exists a sequence of elementary collapses of decreasing dimension from A to a point. Let T be the maximal 1-dimensional subcomplex of K in that sequence. Thus,

$$A \overset{e}{\searrow} A \smallsetminus \{e_1, e_1'\} \overset{e}{\searrow} A \smallsetminus \{e_1, e_1', e_2, e_2'\} \overset{e}{\searrow} \dots \overset{e}{\searrow} A \smallsetminus \{e_1, e_1', e_2, e_2', \dots, e_k, e_k'\} = T \searrow *$$

and T is a spanning tree in K^1. We will see that $\mathcal{P}_{K/A} \sim_{AC} \mathcal{P}_K$, where the latter is constructed using the spanning tree T. Call $A_0 := A$, $A_i := A_{i-1} \smallsetminus \{e_i, e_i'\}$ for $i \geq 1$. We will prove inductively that $\mathcal{P}_{K,A_i} \sim_{AC} \mathcal{P}_{K,A_{i+1}}$. Since $A_{i-1} \overset{e}{\searrow} A_i$, e_i is a free face of e_i' in A_{i-1}. Notice that if

$$\mathcal{P}_{K/A_{i-1}} = \langle x_1, x_2, \dots, x_n \mid r_1, r_2, \dots, r_m \rangle$$

then,

$$\mathcal{P}_{K/A_i} = \langle x_1, x_2, \dots, x_n, e_i \mid e_i^{\epsilon}, e_i^{\epsilon_1} r_1, e_i \epsilon_2 r_2, \dots, e_i^{\epsilon_m} r_m \rangle,$$

where $\epsilon = \pm 1$, depending on the orientation of e_i', and

$$\epsilon_j = \begin{cases} \pm 1 & \text{if } e_i \prec e_j' \text{ (where the sign depends on the orientation of } e_i) \\ 0 & \text{if } e_i \not\prec e_i'. \end{cases}$$

for all $1 \leq j \leq m$. Note that the first relator corresponds to the word spelled on the boundary of e_i'. Now, for every j such that $\epsilon_j \neq 0$, multiply to the left by $e_i^{-\epsilon_j}$ (inverting the relator e_i^{ϵ} if necessary). Finally, invert the relator e_i^{ϵ} if necessary to get the relator e_i and simplify the generator e_i with the relator e_i. As a result of the previous sequence of AC-transformations, we transform $\mathcal{P}_{K/A}$ into $\mathcal{P}_{K/T}$. □

Theorem 1.6. *Let X be a finite poset of height 2 and A be a spanning subdiagram of $\mathcal{H}(X)$. If $\mathcal{K}(A)$ is collapsible, then $\mathcal{P}_{X,A} \sim_{AC} \mathcal{P}_{\mathcal{K}(X)/\mathcal{K}(A)}$.*

Proof. We will exhibit a sequence of AC-moves to transform $\mathcal{P}_{\mathcal{K}(X)/\mathcal{K}(A)}$ into $\mathcal{P}_{X,A}$. The proof is similar to that of Theorem 1.3, and we adopt the notation therein. On the one hand,

$$\mathcal{P}_{X,A} = \langle e_1, \dots, e_r \mid \{(w_{xy}^i)^{-1} w_{xy}^0 : (x,y) \text{ extremal pair of } X, 1 \leq i \leq n_{xy}\}\rangle.$$

On the other hand, since $\mathcal{K}(A)$ is a clique complex, for every extremal pair x, y and every pair of 2-chains $(w_{xy}^i)^{-1}$ and w_{xy}^0, there are two 2-simplices

$$\mathcal{P}_{\mathcal{K}(X),\mathcal{K}(A)} = \langle e_1, \dots, e_r, e_{xy} \mid \{e_{xy}^{-1} w_{xy}^0,\ (w_{xy}^i)^{-1} e_{xy} : (x,y) \text{ extremal pair of } X, 1 \leq i \leq n_{xy}\}\rangle.$$

Call $r_i = (w_{xy}^i)^{-1} * e_{xy}$, $r_0 = e_{xy}^{-1} w_{xy}^0$ and replace in $\mathcal{P}_{\mathcal{K}(X)/\mathcal{K}(A)}$ the relator r_i by $r_i r_0 = (w_{xy}^i)^{-1} w_{xy}^0$, and finally eliminate the generator e_{xy} together with the relator $r_0 = w_{xy}^0 e_{xy}$. □

Theorem 1.7. *Let X be a finite poset of height 2, A a spanning subdiagram of $\mathcal{H}(X)$. If $\mathcal{K}(A)$ is collapsible, then $\mathcal{P}_{X,A} \sim_{AC} \mathcal{P}_{\mathcal{K}(X)}$.*

Proof. By Proposition 1.5 and Theorem 1.6, $\mathcal{P}_{X,A} \sim_{AC} \mathcal{P}_{\mathcal{K}(X)/\mathcal{K}(A)} \sim_{AC} \mathcal{P}_{\mathcal{K}(X)}$. □

Theorem 1.8. *Let $\mathcal{P}$ be a group presentation and A be a spanning collapsible subdiagram of $\mathcal{H}(X_{\mathcal{P}})$. Then $\mathcal{P} \sim_{AC} \mathcal{P}_{X_{\mathcal{P}},A}$.*

Proof. Apply Theorems 1.3 and 1.7 to the poset $X_{\mathcal{P}}$. □

Theorem 1.8 provides a method to obtain AC-equivalent presentations of a given balanced presentation $\mathcal{P}$ by simply choosing an appropriate spanning subdiagram A of $\mathcal{H}(X_{\mathcal{P}})$ and constructing the balanced presentation $\mathcal{P}_{X_{\mathcal{P}},A}$. This procedure could generate more tractable AC-equivalent presentations without specifying the actual AC-transformation. From the proof of Theorem 1.8, the number of movements involved can be estimated as $O(k(n+m))$, where n is the number of generators of $\mathcal{P}$, m the number of relators and k the total relator length. However, it not necessary to explicitly compute them to obtain the desired AC-equivalence.

In Section 3 this result will be used to investigate potential counterexamples to the conjecture.

2. Discrete Morse Theory and deformations

Discrete Morse theory was introduced by Forman as a discrete approach to classical Morse theory. It is built over the notion of a Morse function, which can be thought of as a specific way of labeling the cells of a regular complex which is almost increasing with respect the dimension. In this section we show that Morse theory actually provides a method to 'simplify' the structure of an n-dimensional complex through an $(n+1)$-deformation.

For a comprehensive exposition on discrete Morse Theory and applications, the reader may consult Forman's articles [13, 14], Chari's article [9], and D. Kozlov's book [19]. We briefly recall here the main definitions and results. Let K be a *regular* CW-complex, i.e a CW-complex in which for every open cell e^n, the characteristic map $D^n \to \overline{e^n}$ is a homeomorphism. A map $f : K \to \mathbb{R}$ is a discrete *Morse function* if for every cell e^n in K, the number of faces and cofaces of e^n for which the value of f does not increase with dimension is at least one. An n-cell $e^n \in K^n$ is a *critical cell of index n* if the values of f in every face and coface of e^n increase with dimension. A Morse function induces an ordering in the cells, which determines level subcomplexes of K. For every $c \in \mathbb{R}$, the *level subcomplex $K(c)$ of K* is the subcomplex of closed cells $\bar{e}$ of K such that $f(e) \leq c$. Morse functions serve as a tool to study the homotopy type of K. The following is one of the main results of discrete Morse theory.

Theorem 2.1. [13] *Let K be a regular CW-complex and let $f : K \to \mathbb{R}$ be discrete Morse function. Let $a < b$ be real numbers.*

1. *If the cells e with $f(e) \in (a, b]$ are not critical, then $K(b) \searrow K(a)$.*
2. *If e^n is the only critical cell with $f(e^n) \in (a, b]$, then there is a continuous map $\varphi : \partial D^n \to K(a)$ such that $K(b)$ is homotopy equivalent to $K(a) \cup_\varphi D^n$.*
3. *K is homotopy equivalent to a CW-complex with exactly one cell of dimension k for every critical cell of index k.*

Every discrete Morse function f has an associated set M_f of pairings of cells, where $\{e, e'\} \in M_f$ if and only if $e \leq e'$ and $f(e) \geq f(e')$. A pairing M of cells in K is said to be an *acyclic matching* if each cell of K is involved in at most one pair of M and the directed graph $\mathcal{H}_M(\mathcal{X}(K))$ obtained by reversing the orientation of the edges associated to matched pairs of cells in the Hasse diagram of $\mathcal{X}(K)$ is acyclic. In [9], Chari proved that Morse functions on K are in correspondence with acyclic matchings on $\mathcal{H}(\mathcal{X}(K))$.

We formalize the notion and ideas of *internal collapses* introduced in Kozlov's book [19, Ch 11]. Internal collapses are generalizations of the usual collapses that can be thought of as a simple way of performing an $(n+1)$-deformation from a CW-complex to another one with a *simpler* CW-structure. Internal collapses will be the key to a better understanding of discrete Morse theory and its close connection with simple homotopy theory.

Recall that given K an n-dimensional CW-complex and $\varphi \simeq \psi : S^{n-1} \to K^{n-1}$ attaching maps for the n-cells e^n_φ and e^n_ψ respectively, if $K_\varphi = K \cup e^n_\varphi$ and $K_\psi = K \cup e^n_\psi$, then $K_\varphi \overset{n+1}{\nearrow\searrow} K_\psi$ (see [10, Prop. 7.1]). This is the basic idea behind the internal collapses.

Proposition 2.2. *Let K be a CW-complex of dimension less than or equal to n. Let $\varphi : \partial D^n \to K$ be the attaching map of an n-cell. If $K \searrow L$, then $K \cup_\varphi D^n \overset{n+1}{\nearrow\searrow} L \cup_{\tilde{\varphi}} D^n$, where $\tilde{\varphi} = r\varphi$ and r is the canonical strong deformation retract $r : K \to L$.*

Proof. Let $j : L \to K$ be the inclusion. Then $jr\varphi \cong_H \varphi$ with a homotopy $H : \partial D^n \times I \to K$. We can perform the following sequence of expansions and collapses

$$K \cup_\varphi D^n \nearrow (K \cup_\varphi D^n) \cup_{jr\varphi} D^n \cup_H D^n \times I \searrow K \cup_{jr\varphi} D^n \searrow L \cup_{r\varphi} D^n.$$

Since the dimension of $(K \cup_\varphi D^n) \cup_{jr\varphi} D^n \cup_H D^{n+1}$ is $n+1$, we can conclude that

$$K \cup_\varphi D^n \overset{n+1}{\nearrow\searrow} L \cup_{\tilde{\varphi}} D^n.$$

□

The next result asserts that one can perform a more general procedure of deformation than the described in Proposition 2.2.

Lemma 2.3. *Let $K \cup \bigcup_{i=1}^{N} e_i$ be a CW-complex where K is a subcomplex of dimension at most k and such that $k \leq \dim(e_i) \leq \dim(e_{i+1})$ for all i. Denote by $\varphi_j : \partial D_j \to K \cup \bigcup_{i<j} e_i$ the attaching map of e_j. If $K \searrow L$, then*

$$K \cup \bigcup_{i=1}^{N} e_i \overset{n+1}{\nearrow\searrow} L \cup \bigcup_{i=1}^{N} \tilde{e}_i$$

with $n = \dim(e_N)$ and $\tilde{\varphi}_j : \partial D_j \to L \cup \bigcup_{i<j} \tilde{e}_i$ the attaching map of $\tilde{e}_j$ defined inductively by $\tilde{\varphi}_1 = r\varphi_1$ and if $j > 1$, $\tilde{\varphi}_j = f_j\varphi_j$, where $r : K \to L$ is a strong deformation retraction and $f_j : K \cup \bigcup_{i<j} e_i \to L \cup \bigcup_{i<j} \tilde{e}_i$ is a deformation.

Proof. We proceed by induction in N. If $N = 1$, the assertion follows from Proposition 2.2. Suppose that for every $j \leq N$, there exist deformations $f_j : K \cup \bigcup_{i<j} e_i \to L \cup \bigcup_{i<j} \tilde{e}_i$, defining $\tilde{\varphi}_j = f_j\varphi_j$. By inductive hypothesis, $K \cup \bigcup_{i=1}^{N} e_i \overset{n+1}{\nearrow\searrow} L \cup \bigcup_{i=1}^{N} \tilde{e}_i$ and then, there exists a deformation $f_N : K \cup \bigcup_{i=1}^{N} e_i \to L \cup \bigcup_{i=1}^{N} \tilde{e}_i$. Define $\tilde{\varphi}_{N+1} = f_N\varphi_{N+1}$ and take $L \cup \bigcup_{i=1}^{N+1} \tilde{e}_i$. We

will prove that $K \cup \bigcup_{i=1}^{N+1} e_i \nearrow\!\!\searrow^{n+2} L \cup \bigcup_{i=1}^{N+1} \tilde{e}_i$. In fact,

$$K \cup \bigcup_{i=1}^{N+1} e_i \nearrow K \cup \bigcup_{i=1}^{N+1} e_i \cup \bigcup_{i=1}^{N+1} D_i \cup \bigcup_{i=1}^{N+1} D_i \times I$$

with D_i attached by the map $j_i f_i$ where j_i is a homotopy inverse of f_i, and $D_i \times I$ is attached by the homotopy H_i between $j_i f_i$ and the identity. Now

$$K \cup \bigcup_{i=1}^{N+1} e_i \cup \bigcup_{i=1}^{N+1} D_i \cup \bigcup_{i=1}^{N+1} D_i \times I \searrow K \cup \bigcup_{i=1}^{N+1} D_i \searrow L \cup \bigcup_{i=1}^{N+1} \tilde{e}_i.$$

□

Corollary 2.4. *If K is an n-dimensional CW-complex and L is a collapsible subcomplex of K, then $K \nearrow\!\!\searrow^{n+1} K/L$.*

Definition 2.5. If $K \searrow L$, we say that there is a *internal collapse* from $K \cup \bigcup_{i=1}^{n} e_i$ to $L \cup \bigcup_{i=1}^{n} \tilde{e}_i$, where the cells $\tilde{e}_i$ are attached as described in Theorem 2.3.

By applying successively Theorem 2.3, is easy to see that the composition of successive internal collapses is also an $(n+1)$-deformation.

Theorem 2.6. *Let $L_1 \leq K_1 \leq L_2 \leq K_2 \leq \cdots \leq L_{N-1} \leq K_{N-1} \leq L_N$ be a chain of CW-subcomplexes of L_N such that $K_i \searrow L_i$ for all i. If $L_{i+1} = K_i \cup \bigcup_{j=1}^{k_i} e_j^i$, then*

$$L_N \nearrow\!\!\searrow^{n+1} L_1 \cup \bigcup_{i=1}^{N} \bigcup_{j=1}^{k_i} \tilde{e}_j^i,$$

with $n = \dim(L_N)$.

We will use discrete Morse theory to simplify the cell structure of a CW-complex without changing its *simple* homotopy type. Concretely, any matching in the Hasse diagram of the face poset of a regular CW-complex K can be interpreted as an ordered sequence of internal collapses in its cell structure. Given a poset X and an acyclic matching M in $\mathcal{H}(X)$, it is possible to find a linear extension L of the (order induced by the) directed acyclic graph $\mathcal{H}_M(X)$ respecting the increasing height of critical points such that if $(x, y) \in M$, the points x, y follow consecutively in L (see [19, Thm. 11.2.]). We shall call it a *preferred topological sort* of $\mathcal{H}_M(X)$. The notion of level subcomplex can then be reformulated in terms of the matching. If $L : e_1 \leq e_2 \leq \cdots \leq e_n$ is a preferred topological sort of $\mathcal{H}_M(X)$, then

$$K(c) = \bigcup_{\substack{e_i \in K \\ i \leq c}} \bar{e}_i$$

that is, $K(c)$ is the subcomplex of K generated by the first i cells according to the total order L.

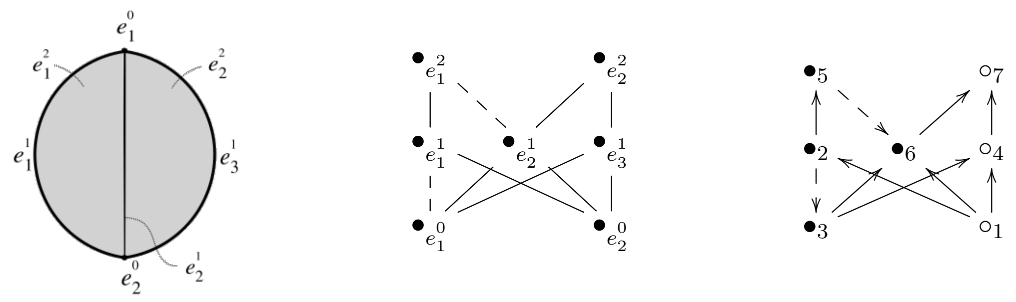

FIGURE 2. A regular 2-complex K, an acyclic matching M and a preferred topological sort L of $\mathcal{H}_M(\mathcal{X}(K))$.

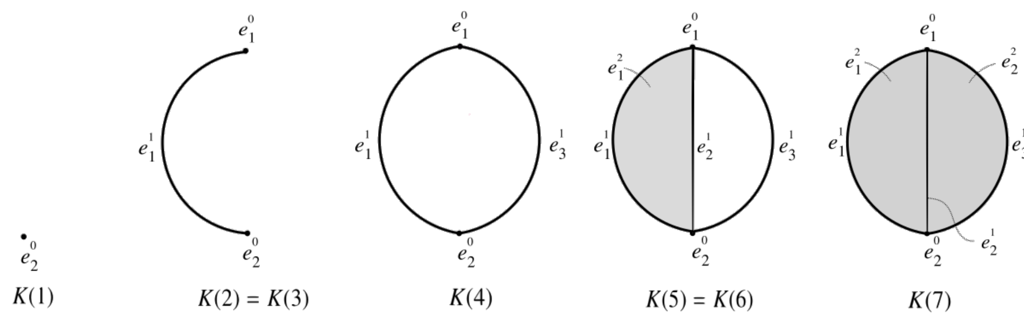

FIGURE 3. The level subcomplexes of K associated to the preferred topological L of Figure 2.

We next state the simple homotopy version of Theorem 2.1, with an explicit construction of the CW-complex equivalent to the original one, and explicit bounds on the deformation. Let K be a regular CW-complex, M an acyclic matching in $\mathcal{X}(K)$ and L a preferred topological sort of $\mathcal{H}_M(\mathcal{X}(K))$. If (e_i, e_{i+1}) is a matched pair of cells, then there is a collapse of the level subcomplexes $K(i+1) \searrow K(i-1)$. Denote by K_M the CW-complex obtained after successively performing the internal collapses determined by the matched pairs of cells.

Theorem 2.7. *Let K be a regular CW-complex of dimension n and let M be an acyclic matching in $\mathcal{H}(\mathcal{X}(K))$. Then $K \overset{n+1}{\nearrow\searrow} K_M$.*

Proof. It is a direct consequence of Theorem 2.1 and Theorem 2.6. □

Example 2.8. Let K be a regular CW-structure of D^1 with two 0-cells e_1^0, e_2^0, three 1-cells e_1^1, e_2^1, e_3^1 and two 2-cells e_1^2, e_2^2 of Figure 2. Let M be the acyclic matching with paired cells (e_1^0, e_1^1) and (e_2^1, e_1^2) and L a preferred topological sort of $\mathcal{H}_M(\mathcal{X}(K))$. Figure 3 describes the successive level subcomplexes, and Figure 4 depicts the 3-deformation from K to K_M, where the latter is the regular CW-structure of D^1 with one 0-cell, one 1-cell and one 2-cell.

Lemma 2.9. *Let K be a regular CW-complex and M be a matching in the subposet of $\mathcal{X}(K)$ of cells of dimension 0 and 1 with only one critical cell of dimension 0. Then M is acyclic if and only if the subcomplex $T = \bigcup_{e \in M} \bar{e}$ of matched cells is a spanning tree in the 1-skeleton of K.*

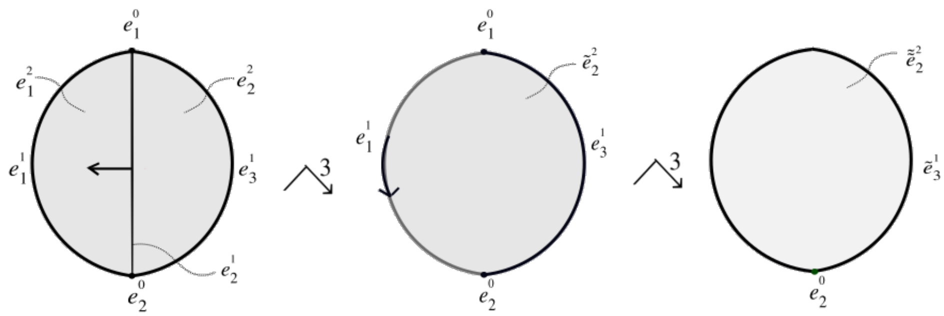

FIGURE 4. The 3-deformation from K to K_M.

Proof. Observe that the subgraph of $\mathcal{H}(\mathcal{X}(K))$ formed by the vertices of height 0 and 1 is the barycentric subdivision of the 1-skeleton of K. A cycle

$$e_1 \prec e'_1 \succ e_2 \prec e'_2 \succ \dots e_n \prec e'_k \succ e_1$$

in $\mathcal{H}_M(\mathcal{X}(K))$, with e_i of height 0 and e'_i of height 1, is in correspondence with a cycle

$$e_1, e_2, \dots, e_k, e_1$$

in the 1-dimensional subcomplex of K of matched cells. The result follows easily from this fact. □

Corollary 2.10. *Let K be a regular CW-complex of dimension n and M a matching in the subposet of $\mathcal{X}(K)$ of cells of dimension 0 and 1 with only one critical cell of dimension 0. Let T be the associated spanning tree in K^1. Then K_M is homeomorphic to K/T and $K \nearrow\searrow^{n+1} K/T$.*

We are particularly interested in the consequences of Theorem 2.7 when the dimension of the complex is 2.

Theorem 2.11. *Let K be a regular CW-complex of dimension 2 and let M be an acyclic matching in $\mathcal{H}(X(K))$. Then $K \nearrow\searrow^{3} K_M$. In particular, $\mathcal{P}_K \sim_{AC} \mathcal{P}_{K_M}$.*

We show now a more concrete and algorithmically manageable version of Theorem 2.11 for group presentations. Recall that if we are given a CW-complex K of dimension 2, the group presentation $\mathcal{P}_K$ associated to K is a presentation of the fundamental group of K/T with T a spanning tree in K^1. By Lemma 2.9 and Corollary 2.10, one can describe $\mathcal{P}_K$ simply in terms of matchings as the presentation of the fundamental group of K_M where M is an acyclic matching whose matched cells are of dimension 0 and 1, and it has only one critical cell of dimension 0. We will provide an easy description of the presentation associated to K_M for a general matching M, which will be more tractable through computer assistance.

Definition 2.12. Let $\mathcal{P}$ be a group presentation and r be a relator of $\mathcal{P}$ given by the word $w_1 x^\epsilon w_2$, where w_1 and w_2 are words on the generators, the generator x appears neither in w_1 nor in w_2, and $\epsilon = +1$ or -1. Then, the *equivalent expression of x inferred by r* is $(w_1^{-1} w_2^{-1})^\epsilon$.

Remark 2.13. If $\mathcal{P}$ is a group presentation and x is a generator of $\mathcal{P}$ such that it appears only once in a relator r, then $\mathcal{P}$ is AC-equivalent to the presentation obtained after eliminating the generator x and the relator r and replacing every occurrence of x in the other relators by its equivalent expression inferred by r. Indeed, suppose r' is another relator containing x. By cyclically permuting r' if necessary, we can assume r' reads as $x^\epsilon u$, with $\epsilon = 1$ or -1. We replace r' by the product sr' where s is a suitable cyclic permutation of r (or its inverse) to eliminate this occurrence of x. We iterate this procedure until no occurrence of x is left.

For instance, if $\mathcal{P} = \langle x, y, z \mid xyx^{-1}y^{-1},\ y^2z^3,\ zxz^{-1}y^{-1}\rangle$, then the equivalent expression of x inferred by $zxz^{-1}y^{-1}$ is $z^{-1}(z^{-1}y^{-1})^{-1}$, i.e., $z^{-1}yz$. Thus, $\mathcal{P}$ is AC-equivalent to the presentation $\tilde{\mathcal{P}} = \langle y, z \mid z^{-1}yzyz^{-1}y^{-1}zy^{-1},\ y^2z^3\rangle$.

Notice that internal collapses transform regular CW-complexes into combinatorial complexes. Recall that a CW-complex K of dimension 2 is called *combinatorial* if for each 2-cell e^2, its attaching map $\varphi : S^1 \to K^{(1)}$ sends each open 1-cell of some CW-structure on S^1 either homeomorphically onto an open 1-cell of K or collapses it to a 0-cell of K (see [16]). Thus, one can think of the attaching map of a 2-cell in a combinatorial complex of dimension 2 simply as the ordered list of oriented 1-cells. Suppose that there is an internal collapse from the combinatorial 2-complex $K \cup \bigcup_{i=1}^n e_i$ to $L \cup \bigcup_{i=1}^n \tilde{e}_i$, where $K = L \cup \{a, e\} \searrow^e L$ and the attaching map of e is, say, $ax_1 \dots x_r$. We endow the CW-complex $L \cup \bigcup_{i=1}^n \tilde{e}_i$ with a combinatorial structure as follows. For every 2-cell containing edge a^ϵ (where, as usual, $\epsilon = 1$ or -1 and by a^{-1} we mean the 1-cell a traversed with the opposite orientation), modify its attaching map by replacing each occurrence of a^ϵ by $(x_1 \dots x_r)^{-\epsilon}$ and leaving the other cells unchanged. Therefore, after performing a sequence of internal collapses to a regular complex we obtain a complex with a natural combinatorial structure.

Definition 2.14. Let K be a regular CW-complex of dimension 2. Let M be an acyclic matching in $\mathcal{H}(\mathcal{X}(K))$ such that there is only one critical cell of dimension 0. Denote by M_0 the subset of matched pairs of cells of dimension 0 and 1, and by $M_r = \{(x_1, y_1), (x_2, y_2), \dots, (x_r, y_r)\}$ the subset of matched pairs of cells of dimension 1 and 2. Let L be a preferred topological sort in $\mathcal{H}_M(\mathcal{X}(K))$. Sort M_r respecting the total order L, that is, $M_r : y_1 < x_1 < y_2 < x_2 < \cdots < y_r < x_r$, with $(x_i, y_i) \in M$. The *group presentation* $\mathcal{Q}_{\mathcal{X}(K),M}$ *associated to the matching* M is the presentation $\mathcal{Q}_r$ defined by the following iterative procedure:

- $\mathcal{Q}_0$ is the standard presentation $\mathcal{P}_K$ constructed using the spanning tree T induced by M_0 (see Lemma 2.9). The generators of Q_0 are the unmatched 1-cells of K according to the matching M_0, and its relators are the words induced by the attaching maps of the 2-cells of K.
- For $0 \leq i < r$, let $\mathcal{Q}_{i+1}$ be the presentation obtained from $\mathcal{Q}_i$ after removing the relator associated to y_{r-i} and the generator x_{r-i}, and replacing every occurrence of the generator x_{r-i} in the remaining relators by the equivalent expression of x_{r-i} inferred by the relator associated to y_{r-i}.

It follows that the group presentation $\mathcal{Q}_{\mathcal{X}(K),M}$ associated to a matching M in $\mathcal{H}(\mathcal{X}(K))$ is AC-equivalent to $\mathcal{P}_K$.

Theorem 2.15. *Let K be a regular CW-complex of dimension 2, and let M be an acyclic matching in $\mathcal{H}(\mathcal{X}(K))$ with only one critical cell of dimension 0. Then, $\mathcal{Q}_{\mathcal{X}(K),M} = \mathcal{P}_{K_M}$ for a suitable choice of orientations and basepoints in K_M.*

Proof. Let $r > 0$. Notice that the combinatorial operation made to get $\mathcal{Q}_1$ from $\mathcal{Q}_0$ parallels exactly the geometric description of an internal collapse provided in the paragraph following Remark 2.13, where the collapse is the one indicated by the pair (x_r, y_r). Since the matching M is acyclic, for every $0 \leq i < r$ the relator corresponding to cell y_{r-i} in $\mathcal{Q}_i$ contains a unique occurrence of generator x_{r-i}. Therefore, presentations $\mathcal{Q}_i$ and $\mathcal{Q}_{i+1}$ are AC-equivalent and the sequence of AC-transformation employed to get from one to the other matches perfectly the corresponding internal collapse in complex K. Denote by K_{M_r} the complex obtained from K after performing the internal collapses induced by the pairs in M_r. Since $K_M = K_{M_r}/T$, it follows that $\mathcal{Q}_r = \mathcal{Q}_{\mathcal{X}(K),M}$ is the standard presentation associated to K_M for the right choice of orientations and basepoints. □

By Theorem 2.11, any regular CW-complex K 3-deforms to K_M. Thus, $Q_{\mathcal{X}(K),M}$ is another representative of the AC-class of $\mathcal{P}_K$.

Corollary 2.16. *Let K be a regular CW-complex of dimension 2, and let M be an acyclic matching in $\mathcal{H}(\mathcal{X}(K))$ with only one critical cell of dimension 0. Then, $\mathcal{Q}_{\mathcal{X}(K),M} \sim_{AC} \mathcal{P}_K$.*

Example 2.17 (The Triangle)**.** Let T be the regular CW-complex with oriented cells of Figure 5, and M be the acyclic matching in $\mathcal{H}(\mathcal{X}(T))$ of Figure 6.

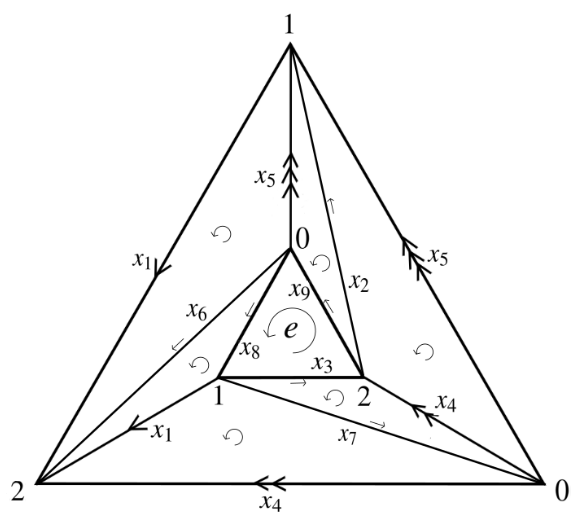

FIGURE 5. The Triangle.

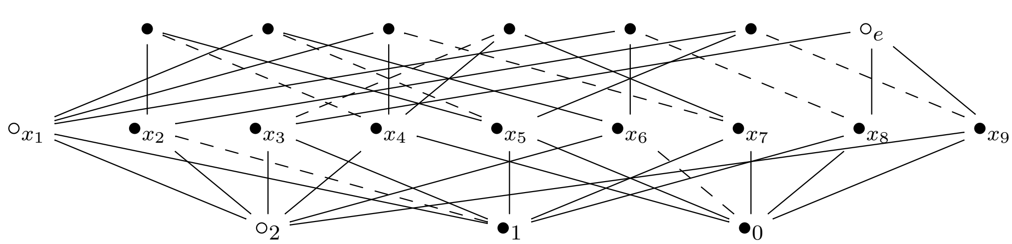

FIGURE 6. An acyclic matching M in $\mathcal{H}(\mathcal{X}(T))$.

The CW-complex T_M has only one 0-cell, one 1-cell x_1 and one 2-cell $\tilde{e}$. However, we do not know beforehand what the attaching map of $\tilde{e}$ looks like. We will prove that $\tilde{e}$ has an attaching map homotopic to x_1^{-1} and so $T_M \nearrow\!\!\searrow^3 D^2$. We take a preferred topological sort of $\mathcal{H}_M(\mathcal{X}(T))$ as in Figure 7.

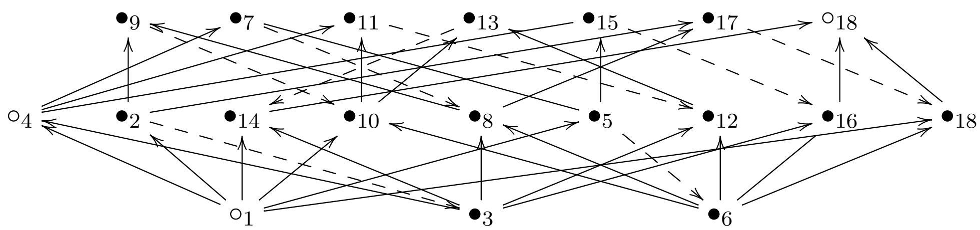

FIGURE 7. Preferred topological sort of $\mathcal{H}_M(T)$.

The attaching map of e in T can be described as the ordered list of 1-cells $x_9x_8x_3$. We carry out explicitly the recursive process described in Theorem 2.15 to obtain the attaching map of $\tilde{e}$ in T_M. By performing the internal collapses indicated by the matched pairs (17,18), (15,16) and (13,14) we get respectively

$$x_9 = x_2x_5^{-1},\ x_8 = x_6x_1^{-1},\ \text{and } x_3 = x_7x_4.$$

From the sequence of internal collapses induced by pairs (11,12), (9,10) and (7,8) we obtain respectively

$$x_7 = x_1x_4^{-1},\ x_4 = x_5x_2^{-1},\ \text{and } x_5 = x_6x_1^{-1},$$

so that $x_4 = x_6x_1^{-1}x_2^{-1}$ and $x_7 = x_1x_2x_1x_6^{-1}$. Thus, the attaching map of $\tilde{e}$ in the combinatorial complex which results from performing the internal collapses that correspond to cells of dimension 1 and 2, turns out to be $x_2x_6x_1^{-1}x_6x_1^{-1}x_1x_2x_1x_6^{-1}x_6x_1^{-1}x_2^{-1}$. The movements induced by the pairs (5,6) and (2,3) amount to the identities

$$x_6 = 1 \text{ and } x_2 = 1.$$

Finally the attaching map of $\tilde{e}$ in T_M is $x^{-1}x_1^{-1}x_1x_1x_1^{-1}$ which is easily seen to be homotopic to x_1^{-1}.

Theorem 2.18. *Let $\mathcal{P}$ be a balanced presentation of a group. Let M be an acyclic matching in $\mathcal{H}(X_{\mathcal{P}})$ with only one critical cell of dimension 0. Then $\mathcal{P} \sim_{AC} \mathcal{Q}_{X_{\mathcal{P}},M}$.*

Proof. Apply Corollary 2.16 to $K'_{\mathcal{P}}$. □

Remark 2.19. Given $\mathcal{P}$ a presentation and M a matching in $\mathcal{H}(X_{\mathcal{P}})$ with only one critical 0-cell, we proved that $\mathcal{P} \sim_{AC} \mathcal{Q}_{X_{\mathcal{P}},M}$. We will estimate the (sufficient) number of AC-transformations to obtain $\mathcal{Q}_{X_{\mathcal{P}},M}$ from $\mathcal{P}$ Theorem 2.18 follows from the chain of equivalences:

$$\mathcal{P} \sim_{AC} \mathcal{P}_{K_{\mathcal{P}}} \sim_{AC} \mathcal{P}_{K'_{\mathcal{P}}} \sim_{AC} \mathcal{P}_{(K'_{\mathcal{P}})_M} = \mathcal{Q}_{X_{\mathcal{P}},M}.$$

Let n be the number of generators of $\mathcal{P}$, m the number of relators and k the total relator length. The equivalences $\mathcal{P} \sim_{AC} \mathcal{P}_{K_{\mathcal{P}}}$ and $\mathcal{P}_{K_{\mathcal{P}}} \sim_{AC} \mathcal{P}_{K'_{\mathcal{P}}}$ can be achieved in $O(n+m)$ and $O(k)$ AC-transformations respectively. Now, by Theorem 2.11, $K'_{\mathcal{P}} \nearrow\!\!\searrow^3 (K'_{\mathcal{P}})_M$. Thus, the estimated number of AC-transformations required to obtain $\mathcal{P}_{(K'_{\mathcal{P}})_M}$ from $\mathcal{P}_{K'_{\mathcal{P}}}$ has the order of the number of elementary expansions and collapses needed to deform $K'_{\mathcal{P}}$ into $(K'_{\mathcal{P}})_M$. This is bounded by the square of the number of cells of $K'_{\mathcal{P}}$, which is proportional to k.

3. Applications to potential counterexamples

Over the last fifty years a list of examples of balanced presentations of the trivial group which are not known to be trivializable via Andrews-Curtis transformations has been compiled. They serve as *potential counterexamples* to disprove the conjecture (see [17]).

The list of the potential counterexamples that we are going to consider is shown below. See [17] for an extended list and [26] for a recent overview.

(1) $\mathcal{AK}_n = \langle x, y \mid xyx = yxy, x^n = y^{n+1}\rangle$, with $n \geq 2$. Akbulut and Kirby [2].
(2) $\mathcal{R} = \langle x, y, z \mid z^{-1}yz = y^2, x^{-1}zx = z^2, y^{-1}xy = x^2\rangle$. Rapaport [23].
(3) $\mathcal{G}_{n,m,p,q} = \langle x, y \mid x = [x^m, y^n], y = [x^p, y^q]\rangle$, with $n, m, p, q \in \mathbb{Z}$. Gordon [8].

We expose some practical results achieved after implementing the algorithms developed in Sections 1 and 2 in the SAGE platform [25]. The proof of these results, as well as the Python code of the implementation can be found in [12].

Given $\mathcal{P}$ a potential counterexample, the general outline to prove that $\mathcal{P} \sim_{AC} \langle\ \mid\ \rangle$ is to find an appropriated matching or subdiagram of $\mathcal{H}(X_{\mathcal{P}})$ and then obtain a new presentation in the same AC-class, but computationally tractable.

We say that a presentation is greedily AC-trivializable if the (greedy) algorithm of simplification of presentations described in [15] (and also implemented in SAGE) can transform it into $\langle\ \mid\ \rangle$. This procedure was originally thought for the Tietze-simplification of presentations and it consists in a loop of two phases. The *search phase* attempts to reduce the length-relator by replacing long substrings of relators by shorter equivalent ones. That is, if there are relators r_1 and r_2 such that a suitable cyclic permutation of r_1 reads uv and a cyclic permutation of r_2, or its inverse reads as wv, and the length of v is greater than the length of w; then r_2 is replaced by wu^{-1}. The *elimination phase* tries to eliminate generators which occur only once in some relator. The previous algorithm actually makes only AC-transformations if we start from a balanced presentation of the trivial group. In fact, there is only one situation in which the procedure makes a transformation not included in (1)-(4). Suppose that the presentation $\mathcal{P}$ has a relator r_i which is equal to another relator r_j. Then, this algorithm replaces relator r_i by a 1, and then eliminates the latter 1. This transformation changes the deficiency of the presentation and does not preserve the (simple) homotopy type of the associated complex $K_{\mathcal{P}}$. However, the previous situation is not possible if $\mathcal{P}$ is a balanced presentation of the trivial group, since if $\mathcal{P}$ has (after possibly a sequence of AC-transformations) one relator equal to another, then it is in the same AC-class as a presentation with a relator equal to 1. Thus, $K_{\mathcal{P}}$ has non-trivial second homology group, which is not possible.

We present the following results.

- The presentation $\mathcal{AK}_2 = \langle x, y \mid xyx = yxy, x^2 = y^3\rangle$ satisfies the Andrews-Curtis conjecture.

  We found a collapsible subdiagram A in $\mathcal{H}(X_{\mathcal{AK}_2})$ such that $\mathcal{P}_{X_{\mathcal{AK}_2},A}$ is greedily AC-trivializable. Since $\mathcal{AK}_2 \sim_{AC} \mathcal{P}_{X_{\mathcal{AK}_2},A}$, this proves that $\mathcal{AK}_2$ satisfies the

Andrews-Curtis conjecture. This fact was first proved in [20] using genetic algorithms. For $n = 1$, $\mathcal{AK}_1$ can be trivially AC-transformed into $\langle \mid \rangle$; for $n > 2$, the question remains open.

- The presentation $\mathcal{R}$ posed by Rapaport can be AC-transformed into a presentation with 2 generators and 2, and quite reduced total length-relator.

  We found a collapsible subdiagram A in $\mathcal{H}(X_{\mathcal{R}})$ for which $\mathcal{P}_{X_{\mathcal{R}},A}$ can be greedily transformed into

$$\tilde{\mathcal{R}} = \langle x, y \mid x^{-1}yx^{-2}y^{-1}xyx^2y^{-1}x^{-1}yx^2y^{-1},\ y^{-1}x^{-1}yx^2y^{-2}x^{-1}yx^2yx^{-2}y^{-1}x\rangle,$$

  whose relators have total length equal to 31. $\mathcal{R}$ is the only potential counterexample with 3 generators and relators. This transformation may be useful given the extensive study of the class of '2-relators' presentations (see [6, 18, 21, 22]).
- The presentations of the family

$$\mathcal{G}_{1,1,k,-1} = \langle x, y \mid x = [x^{-1}, y^{-1}], y = [x^{-k}, y]\rangle,\ \text{with } 1 < k < 100,$$

  satisfies the Andrews-Curtis conjecture.

  It is not known in general if the family of presentations $\mathcal{G}_{n,m,p,q}$ are AC-trivializable. The best previous result was obtained in [6], where the authors proved that the presentations in this sequence whose total length-relator is up to 14 are can be transformed into $\langle \mid \rangle$ with AC-moves. We focused our attention in the subfamily

$$\mathcal{G}_{1,1,k,-1} = \langle x, y \mid x = [x^{-1}, y^{-1}], y = [x^{-k}, y]\rangle.$$

  If k is even, an easy computation by hand reveals that $\mathcal{G}_{1,1,k,-1} \sim_{AC} \langle \mid \rangle$. Nevertheless, if k is odd, the question was open. For each $1 < k < 100$, k odd, we found an acyclic matching M_k in $\mathcal{H}(X_{\mathcal{G}_{1,1,k,-1}})$ such that $\mathcal{Q}_{X_{\mathcal{G}_{1,1,k,-1}},M_k}$ can be reduced to $\langle \mid \rangle$ after applying the greedy algorithm of trivialization.
- Recently, Barmak proved that a strong formulation of the Andrews-Curtis conjecture is false [4]. He exhibited an example of two presentations which are not Q^*-equivalents even though their standard complexes have the same simple homotopy type. More specifically, he proved that

$$\mathcal{B} = \langle x, y \mid [x, y], 1\rangle$$

  cannot be transformed into

$$\mathcal{B}' = \langle x, y \mid [x, [x, y^{-1}]]^2y[y^{-1}, x]y^{-1}, [x, [[y^{-1}, x], x]]\rangle$$

  using operations (1)-(3) and (5), but $\mathcal{K}_{\mathcal{B}} \diagup\!\!\searrow \mathcal{K}_{\mathcal{B}'}$. He asked whether these presentations are Q^{**}-equivalent or not (being a potential counterexample to the generalized Andrews-Curtis conjecture, see [16, Section 4.1]).

  We proved that $\mathcal{B}$ and $\mathcal{B}'$ are AC-equivalent and thus, Q^{**}-equivalent. Concretely, we found an acyclic matching M in $\mathcal{H}(X_{\mathcal{B}'})$ with only one critical cell in dimension 0 and then greedily simplified $\mathcal{Q}_{X_{\mathcal{P}'},M}$ into $\mathcal{P}$ using operations (1)-(4). Therefore, this counterexample to the strong formulation of the AC-conjecture is not a counterexample to the generalized Andrews-Curtis conjecture.

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